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RELATIONS AMONG SOME CYCLOTOMIC CUBICS

BY T. HAYASHI

1. According to Mr. E. B. Escott's valuable memoir "Cubic congruences with three real roots", published in the *Annals of Mathematics*, second series, vol. 2, pp. 86-92, January, 1910, we have the theorem:

The congruence

$$x^3 + ax^2 + bx + c \equiv 0 \pmod{p}$$

has three roots (when it has any), the relations between the roots being

$$\left. \begin{aligned} \beta &= \alpha^2 + ka + l, \\ \gamma &= \beta^2 + k\beta + l, \\ a &= \gamma^2 + k\gamma + l, \end{aligned} \right\} \quad (1)$$

where

$$b = -(\alpha^2 - 4ak - 2a + 3k^2 + 3k + 3),$$

$$c = -(\alpha^3 - 4a^2k - 2a^2 + 5ak^2 + 5ak + 3a - 2k^3 - 3k^2 - 3k - 1),$$

$$l = -(\alpha^2 - 3ak - a + 2k^2 + 2k + 2).$$

This theorem can be applied to the cubic equation

$$x^3 + ax^2 + bx + c = 0 \quad (2)$$

having three real roots. As Mr. Escott remarks, the cubic equation

$$x^3 + a'x^2 + b'x + c' = 0 \quad (3)$$

which is obtained when we replace a by $-a + 3k + 1$, has three real roots among which the relations (1) exist.

The two cubics (2) and (3) may be said to be *conjugate* with respect to the relations (1).

Using this fact, I will show that the cyclotomic cubics* in the cases of the circle-division into seven, thirteen, and nineteen parts are all conjugate to that in the case of the division into nine parts.

2. The cyclotomic cubic

$$x^3 + x^2 - 2x - 1 = 0$$

whose roots are

* I will use this name for any cubic which can be used for the circle-division.

$$2 \cos \frac{2\pi}{7}, \quad 2 \cos \frac{4\pi}{7}, \quad 2 \cos \frac{6\pi}{7},$$

is obtained by putting $a = 1$, $k = 0$.

If we replace a by $-a + 3k + 1$, i.e. 0, and therefore if we put $a = 0$, $k = 0$, then we get the cubic equation

$$x^3 - 3x + 1 = 0.$$

This cubic is the cyclotomic equation in the case of the circle-division into nine parts, because its roots are

$$2 \cos \frac{2\pi}{9}, \quad 2 \cos \frac{4\pi}{9}, \quad 2 \cos \frac{8\pi}{9}.$$

Therefore *the cyclotomic cubic $x^3 + x^2 - 2x - 1 = 0$ in the case of the circle-division into seven parts is conjugate to the cyclotomic cubic $x^3 - 3x + 1 = 0$ in the case of the circle-division into nine parts with respect to*

$$\left. \begin{aligned} \beta &= a^2 - 2, \\ \gamma &= \beta^2 - 2, \\ a &= \gamma^2 - 2. \end{aligned} \right\}$$

If

$$a^3 + a^2 - 2a - 1 = 0,$$

$$\beta = a^2 - 2 = -\frac{1}{a+1}.$$

Similarly

$$\gamma = \beta^2 - 2 = -\frac{1}{\beta+1},$$

$$a = \gamma^2 - 2 = -\frac{1}{\gamma+1}.$$

Again if

$$a^3 - 3a + 1 = 0,$$

$$\beta = a^2 - 2 = \frac{a-1}{a}.$$

Changing the signs of a and β , and transforming,

$$a = -\frac{1}{\beta+1}.$$

Hence if a, β, γ be the roots

$$-2 \cos \frac{2\pi}{9}, \quad -2 \cos \frac{4\pi}{9}, \quad -2 \cos \frac{8\pi}{9}$$

of the cubic equation

$$x^3 - 3x - 1 = 0,$$

then

$$\begin{aligned} a &= -\frac{1}{\beta + 1}, \\ \beta &= -\frac{1}{\gamma + 1}, \\ \gamma &= -\frac{1}{a + 1}. \end{aligned}$$

Therefore by changing the order of the roots, we know that *the cyclotomic cubic $x^3 + x^2 - 2x - 1 = 0$ in the case of the circle-division into seven parts is conjugate to the cyclotomic cubic $x^3 - 3x - 1 = 0$ in the case of the division into nine parts with respect to*

$$\left. \begin{aligned} \beta &= -\frac{1}{a + 1}, \\ \gamma &= -\frac{1}{\beta + 1}, \\ a &= -\frac{1}{\gamma + 1}. \end{aligned} \right\}$$

3. The cyclotomic cubic

$$x^3 + x^2 - 4x + 1 = 0$$

whose roots are

$$2 \cos \frac{2\pi}{13} + 2 \cos \frac{10\pi}{13}, \quad 2 \cos \frac{4\pi}{13} + 2 \cos \frac{6\pi}{13}, \quad 2 \cos \frac{8\pi}{13} + 2 \cos \frac{12\pi}{13},$$

is obtained by putting $a = 1$, $k = 1$.

If we replace a by $-a + 3k + 1$, i.e. 3, and therefore if we put $a = 3$, $k = 1$, we get the cubic equation

$$x^3 + 3x^2 - 3 = 0$$

whose roots are

$$-2 \cos \frac{2\pi}{9} - 1, \quad -2 \cos \frac{4\pi}{9} - 1, \quad -2 \cos \frac{8\pi}{9} - 1.$$

Therefore *the cyclotomic cubic $x^3 + x^2 - 4x + 1 = 0$ in the case of the circle-division into thirteen parts is conjugate to the cyclotomic cubic $x^3 + 3x^2 - 3 = 0$ in the case of the circle-division into nine parts with respect to*

$$\left. \begin{aligned} \beta &= a^2 + a - 3, \\ \gamma &= \beta^2 + \beta - 3, \\ a &= \gamma^2 + \gamma - 3. \end{aligned} \right\}$$

We can also show that *the cyclotomic cubic $x^3 + x^2 - 4x + 1 = 0$ in the case of the circle-division into thirteen parts is conjugate to the cyclotomic cubic $x^3 - 3x + 1 = 0$ in the case of the circle-division into nine parts with respect to*

$$\left. \begin{aligned} \beta &= \frac{a-1}{a}, \\ \gamma &= \frac{\beta-1}{\beta}, \\ a &= \frac{\gamma-1}{\gamma}. \end{aligned} \right\}$$

4. The cyclotomic cubic in the case of the circle-division into nineteen parts is

$$x^3 + x^2 - 6x - 7 = 0.$$

(Gauss, *Werke*, vol. 1, § 353.) This is obtained by putting $a = 1$, $k = -1$.

If we replace a by $-a + 3k + 1$, i.e. -3 , and therefore if we put $a = -3$, $k = -1$, we get the cubic equation

$$x^3 - 3x^2 - 6x + 17 = 0.$$

Solving this cubic by Trigonometry, we get as its roots

$$2\sqrt{3} \cos \frac{5\pi}{18} + 1, \quad 2\sqrt{3} \cos \frac{7\pi}{18} + 1, \quad 2\sqrt{3} \cos \frac{17\pi}{18} + 1,$$

i.e.

$$2 \cos \frac{\pi}{9} + 2 \cos \frac{4\pi}{9} + 1, \quad 2 \cos \frac{2\pi}{9} + 2 \cos \frac{5\pi}{9} + 1, \quad 2 \cos \frac{7\pi}{9} + 2 \cos \frac{10\pi}{9} + 1;$$

so that this cubic may be said to be the cyclotomic cubic in the case of the division into nine parts.

Therefore *the cyclotomic cubic $x^3 + x^2 - 6x - 7 = 0$ in the case of the circle-division into nineteen parts is conjugate also to the cyclotomic cubic $x^3 - 3x^2 - 6x + 17 = 0$ in the case of the division into nine parts with respect to*

$$\left. \begin{aligned} \beta &= a^2 - a - 5, \\ \gamma &= \beta^2 - \beta - 5, \\ a &= \gamma^2 - \gamma - 5. \end{aligned} \right\}$$